HOUSTON JOURNAL OF MATHEMATICS, Volume 11, No. 4, 1985.

MAZUR MANIFOLDS AND WRAPPING NUMBERS OF KNOTS IN $S^1 \times S^2$

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The first example of a contractible 4-manifold other than B^4 was constructed by Mazur [4]. That example was described as the union of $S^1 \times B^3$ and $B^2 \times B^2$ (the "2-handle") identifying the neighborhood of a knot K in $\partial(S^1 \times B^3) = S^1 \times S^2$ with $S^1 \times B^2 \subseteq \partial(B^2 \times B^2) = S^1 \times B^2 \cup B^2 \times S^1$. The identification takes place via the "attaching map." Any manifold constructed in this way will be contractible as long as K represents a generator of $H_1(S^1 \times B^3;Z)$.

Contractible 4-manifolds constructed by the above procedure are now referred to as Mazur manifolds and have been extensively studied. In particular, Akbulut and Kirby [2] proved that certain apparently distinct Mazur manifolds were in fact diffeomorphic. In the final section of this work it is shown that for the examples in [2] the attaching maps are in fact the same up to diffeomorphism. This observation prompted the question of whether or not a single Mazur manifold could be constructed from $S^1 \times B^3$ by attaching the 2-handle in two distinct ways. This paper provides such an example of a pair of diffeomorphic Mazur manifolds whose attaching maps are distinct up to diffeomorphism.

Distinguishing the attaching maps mentioned above leads to an investigation of knots in $S^1 \times S^2$. Sections 2 and 3 of this paper present results concerning the "geometric wrapping number" of knots in $S^1 \times S^2$. In Section 4 some of these results are applied to distinguish the attaching maps for the Mazur manifold constructed in Section 1.

NOTATION. Throughout this paper the notation of [1] and [2] is used. In particular framed link diagrams are used to describe 4-manifolds. Our reference for basic knot theory is [5] and the details needed for cut and paste arguments in

*Supported in part by a grant from the National Science Foundation.

3-manifolds can be found in [3]. All manifolds and maps are smooth.

Section 1. Description of M^4. The first illustration in Figure 1 gives a description of a Mazur manifold, M^4. K is the knot in $S^1 \times S^2$ along which the 2-handle is attached. The diagrams that follow represent alternative handlebody descriptions of M^4. In the final illustration M^4 again appears as a Mazur manifold, in this case with attaching circle J, apparently distinct from K. In Section 4 it is proved that J and K are distinct knots in $S^1 \times S^2$.

REMARK. Before discovering the alterations illustrated in Figure 1, a different method was used in proving that the initial and final manifolds illustrated were diffeomorphic. The proof then used is interesting and will now be sketched. Consider the link of two components in S^3 formed by the two circles in the first illustration. (They indicate the 1 and 2-handles in that diagram.) One can check that this is the same link as in the final illustration. Attach two 2-handles to B^4 along that link, each with 0 framing. As each circle is unknotted in S^3, surgery can be performed on either 2-handle: that is, a regular neighborhood of the 2-sphere formed as the union of the core of the 2-handle and the obvious disk in B^4 bounded by the attaching circles is removed, and replaced with $S^1 \times B^3$.

In one case this surgery results in the manifold of the first illustration, the other possible surgery gives the manifold of the final illustration. It is rather simple to show that the manifolds are diffeomorphic if the 2-spheres along which each surgery is performed can be isotoped to intersect in exactly one point. However, the link being considered is concordant to the Hopf link. Hence, the two classes being surgered can be represented by 2-spheres intersecting precisely once. The proof that the manifolds are diffeomorphic is then completed by showing that the original two 2-spheres along which surgery was performed are isotopic to this new pair.

Section 2. Wrapping numbers of knots in $S^1 \times S^2$. The wrapping number of a knot in $S^1 \times B^2$ is a standard, well known invariant. In this section we discuss the similar concept for knots in $S^1 \times S^2$ and prove a theorem which greatly simplifies the calculation of wrapping number. The results here are related to those of [6] concerning calculations in $S^1 \times B^2$. The added complexity here is based on the fact that $S^1 \times S^2$ is not irreducible, a fact that complicates the cut and paste arguments

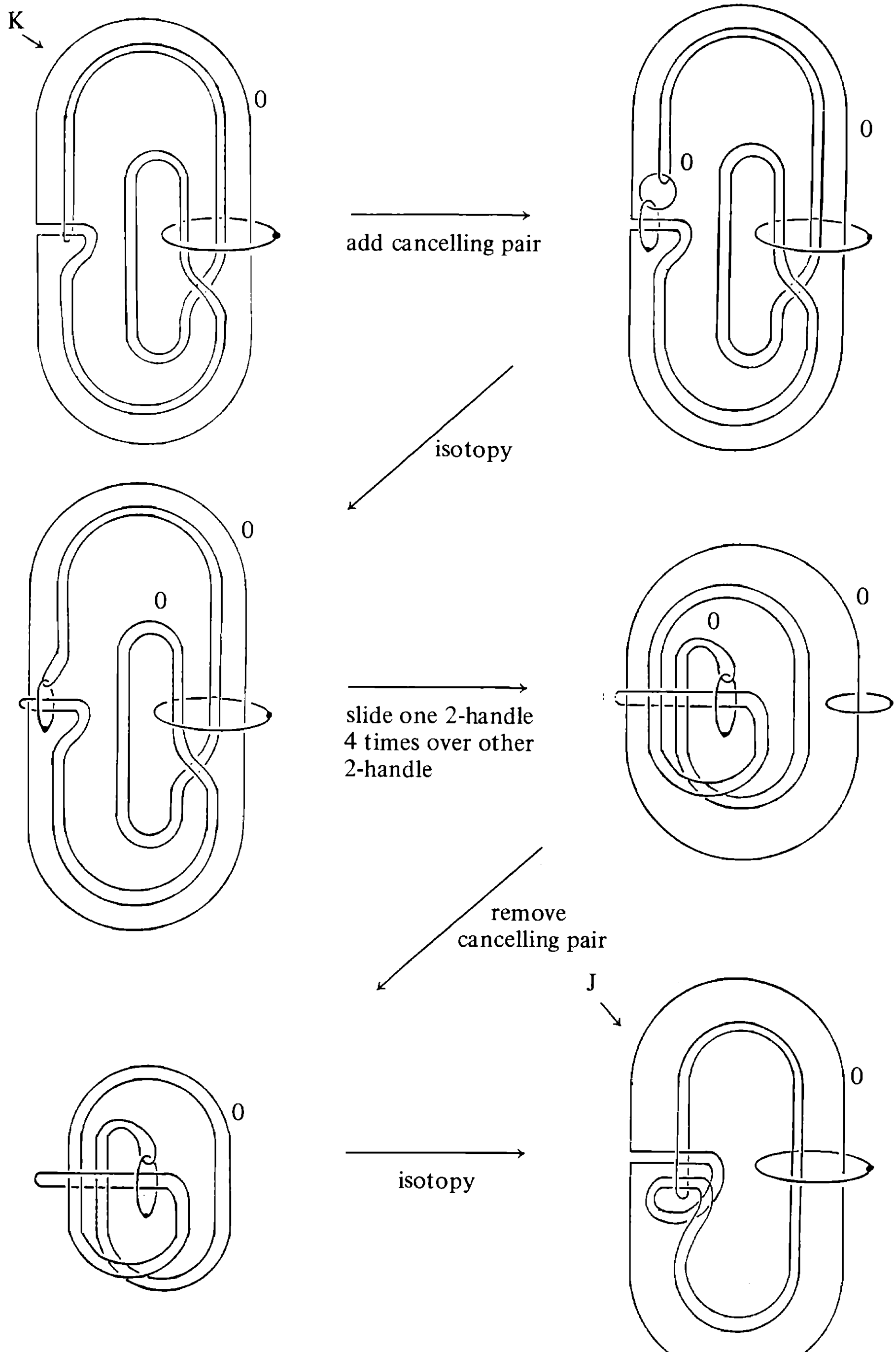
K
0
add cancelling pair
0
0
isotopy
0
0
slide one 2-handle
4 times over other
2-handle
0
0
remove
cancelling pair
J
0
isotopy
0

Figure 1.

needed.

In $S^1 \times S^2$ there are only two isotopy classes of embedded 2-spheres, if we ignore orientations. One bounds a 3-ball, the other is essential. They can be distinguished by their images in $H_2(S^1 \times S^2;Z)$. Any essential 2-sphere is isotopic to $\{pt\} \times S^2$, and generates the second homology group.

DEFINITION. The wrapping number of a knot L in $S^1 \times S^2$ is the minimum number of intersections of L with any essential 2-sphere transverse to L.

If S is an embedded 2-sphere in $S^1 \times S^2$ and D is an embedded disk in $S^1 \times S^2$ with $D \cap S = \partial D$, then surgery can be performed on S using D as follows. ∂D bounds two disks, D_1 and D_2, on S. $D_1 \cup D = S_1$ and $D_2 \cup D = S_2$ are both embedded 2-spheres in $S^1 \times S^2$. After a small isotopy it can be arranged that they are smoothly embedded and disjoint.

LEMMA. *If* S *is an essential embedded* 2*-sphere in* $S^1 \times S^2$ *and* D *satisfies the conditions necessary to do surgery on* S, *then one of* S_1 *and* S_2 *is essential.*

PROOF. It is clear from the construction that the class of S in $H_2(S^1 \times S^2;Z)$ is represented by the sum of S_1 and S_2. From this it is immediate that one of the two is essential if S is.

THEOREM. *Let* K *be a knot in* $S^1 \times S^2$, *transverse to an essential* 2*-sphere,* S, *with* $K \cap S = n$ *points. If there is an essential* 2*-sphere* T *with* $T \cap K < n$, *then there is an essential* 2*-sphere* T′ *disjoint from* S *with* $T' \cap K < n$ *points.*

PROOF. Pick an essential 2-sphere T with $T \cap K < n$ and with $T \cap S$ a minimal number of circles. Assume that $T \cap S \neq \phi$. Consider the following two statements:

P(m): Any disk on S bounded by a circle of intersection with T intersects K in at least m + 1 points.

Q(m): Any disk on T bounded by a circle of intersection with S intersects K in at least m + 1 points.

Notice that P(n) is true only if $S \cap T = \phi$. We will prove P(0), that P(m) implies Q(m), and that Q(m) implies P(m + 1). The theorem then follows.

P(0): If P(0) is false, some circle on S bounds a disk D disjoint from K. Pick an innermost such disk. Use D to perform surgery on T to get T_1 and T_2. $T_1 \cap K$ and $T_2 \cap K$ clearly contain fewer than n points. After a small isotopy, $T_1 \cap S$ and $T_2 \cap S$

contain fewer circles of intersection than $T \cap S$. Yet by the lemma, one of T_1 and T_2 is essential, contradicting the minimality condition on T.

$P(m) \Rightarrow Q(m)$: If Q(m) is false some circle of intersection of T with S bounds a disk D on T intersecting K in m or fewer points. Let D represent an innermost such disk. Perform surgery on S using D constructing spheres S_1 and S_2. After a small isotopy, both can be made disjoint from S. By the lemma, one of the two, say S_1, is essential. It is constructed from S by removing a disk with at least m + 1 points of intersection with K (by P(m)) and replacing it with a disk with at most m points of intersection. Hence $S_1 \cap K < n$ and $S_1 \cap S = \phi$, contradicting the minimality condition on T.

$Q(m) \Rightarrow P(m + 1)$: If $P(m + 1)$ is false, some circle on S bounds a disk D intersecting K in m + 1 or fewer points. Let D be an innermost such disk. The corresponding circle on T bounds disks, each of which intersects K in at least m + 1 points (by Q(m)). Perform surgery on T using D to construct T_1 and T_2. Assume T_1 is essential. T_1 is constructed from T by removing a disk with at least m + 1 points of intersection with K and replacing it with one having at most m + 1 points of intersection with K. Hence $T_1 \cap K \leqslant T \cap K$. After a small isotopy $T_1 \cap S$ has fewer circles of intersection with $T \cap S$, again contradicting the minimality of $T \cap S$.

We will present an application of this result in Section 4.

Section 3. Wrapping numbers and companionship in $S^1 \times S^2$. Schubert [6] proved that if K is a knot in a solid torus $V = S^1 \times B^2$, which is in turn embedded in a solid torus V′, then the wrapping number of K in V′ is the product of the wrapping number of K in V with the wrapping number of the core of V in V′. Here we prove the analagous result for knots in $S^1 \times S^2$.

First the notation: J will be a knot in $S^1 \times S^2$, and λ is the curve $S^1 \times \{pt\} \subseteq S^1 \times S^2$, λ and J are assumed to be disjoint. K is a knot in $S^1 \times B^2 = V$, and $\partial V = T$. C is the core, $S^1 \times \{0\} \subseteq S^1 \times B^2$. f is an embedding of V into $S^1 \times S^2$ such that $f(C) = J$ and $f(V) \cap \lambda = \phi$. The wrapping number of K in V is the minimal number of points of intersection of K with a spanning disk of V. N(J) represents a small regular neighborhood of J, and $\mathring{N}(J)$ the interior of that neighborhood.

THEOREM. *If* $\partial(S^1 \times S^2 - \mathring{N}(J))$ *is incompressible in* $S^1 \times S^2 - \mathring{N}(J)$ *then the* *wrapping number of* f(K) *in* $S^1 \times S^2$ *is the product of the wrapping number of* J *in* $S^1 \times S^2$ *with the wrapping number of* K *in* V.

PROOF. Let S be an essential 2-sphere in $S^1 \times S^2$ intersecting f(K) in the least number of points. Set $n = S \cap f(K)$. Denoting the wrapping number of K in V by j, and the wrapping number of J in $S^1 \times S^2$ by m, it is immediate that $n \leqslant m \cdot j$. We prove $n \geqslant m.j$.

S can be assumed to be transverse to f(T). We proceed by modifying S to eliminate any trivial circles of intersection on f(T). If such a circle of intersection exists, an innermost such circle bounds a disk D on f(T). Use D to perform surgery on S, which results on 2-spheres S_1 and S_2. Since S is essential, one of S_1 and S_2 is. Say it is S_1. In addition, $S_1 \cap f(K) = n$, since by construction $S_1 \cap f(K) \leqslant n$, and n was assumed minimal. $S_1 \cap f(T)$ has fewer trivial circles than $S \cap f(T)$. Repeating this procedure we can find an essential 2-sphere S′ which intersects f(K) in n points and such that $S' \cap f(T)$ contains no trivial circles of intersection on f(T). (Note S′ and S are isotopic.) We now denote S′ by S.

All circles of intersection of S and f(T) are parallel on f(T). We next observe that in fact they are all meridians to f(V). To see this, consider an innermost circle of intersection of $S \cap f(T)$, on S. As $\partial(S^1 \times S^2 - \mathring{N}(J))$ is incompressible, it follows immediately that the disk bounded by that circle on S is contained in f(V), and hence the boundary circle is a meridian.

In the case that $S \cap f(T) = \phi$ we are essentially done. As S is essential it cannot be contained in f(V) and hence $S \cap J = \phi$, so $m = 0$. It is then immediate that $n \geqslant m \cdot j$.

In the case that $S \cap f(T)$ is nonempty, the collection of circles of intersection separate S into connected regions $\{R_i\}_{i=1,\ldots,r}$ and $\{Q_i\}_{i=1,\ldots,q}$, distinguished by the requirement that $Q_i \subseteq f(V)$ and $R_i \cap f(V) = \phi$, for all i. Notice that $\lambda \cap Q_i = \phi$ for all i. Hence all intersections of λ with S are contained in the R_i. As S is essential, $\lambda \cap S$ contains an odd number of points. Hence for some R_i, call it R, $\lambda \cap R$ is an odd number of points.

Next observe that $\partial(\overline{R})$ has at least m components. To prove this, cap off each boundary component of $\overline{R}$ with a meridinal disk in f(V). Call the resulting 2-sphere S_1. As $S_1 \cap \lambda$ is an odd number of points, S_1 is clearly essential. Hence $S_1 \cap J$, which is just the number of boundary components of $\bar{R}$, must be at least the wrapping number of J, m.

The boundary components of $\bar{R}$ bound disjoint disks on S. For each of those disks pick an innermost circle of intersection with f(T). (These circles might not be boundary components of $\bar{R}$.) Each disk on S bounded by such an innermost circle is a meridinal disk to f(V) and hence intersects f(K) in at least j points. Since there are at least m such disks, the theorem follows.

To complete this section we make the following observations:

(1) If J represents $n \in H_1(S^1 \times S^2;Z)$ with $|n| \geqslant 2$ then $S^1 \times S^2 - \mathring{N}(J)$ has incompressible boundary, and hence the theorem applies. To see this, note the kernel of the map $H_1(\partial N(J)) \to H_1(S^1 \times S^2 - \mathring{N}(J))$ is generated by $n \cdot \mu$ where μ is a meridian to J. If $|n| \geqslant 2$ this cannot be represented by an embedded curve on $\partial N(J)$.

(2) If J represents 0 in $H_1(S^1 \times S^2;Z)$ the theorem is again true without conditions on incompressibility. This follows from the observation that in this case if the boundary is compressible, then the wrapping number of J is also 0, in which case the theorem is trivial.

(3) Finally if F represents 1 in $H_1(S^1 \times S^2;Z)$ and the boundary is compressible, then the conclusion of the theorem need not hold. In such a situation J is isotopic to $S^1 \times \{pt\}$ in $S^1 \times S^2$. In Figure 2 a knot, K, in $S^1 \times B^2$ is illustrated. In $S^1 \times B^2$ the knot has wrapping number 3. To prove this, a proof similar to that of the theorem in Section 2 can be used to show: If there is a meridinal disk in $S^1 \times B^2$ intersecting K in 1 point, then there is such a meridinal disk disjoint from the disk D in Figure 2. This result along with an argument similar to that of the next section is sufficient to compute the wrapping number.

If $S^1 \times B^2$ is embedded in $S^1 \times S^2$ with $S^1 \times \{pt\} \to S^1 \times \{pt\}$, then the image of K in $S^1 \times S^2$ is isotopic to $S^1 \times \{pt\}$. Such an isotopy is described in Figure 3. Another example of this "non-multiplicity of wrapping number" can be easily constructed from the examples on page 104 of [1].

Section 4. The wrapping numbers of J and K. Consider the knot K in $S^1 \times S^2$ illustrated in Figure 1. We will now show it has wrapping number 5 in $S^1 \times S^2$. A similar demonstration that J has wrapping number 3 will be left to the reader.

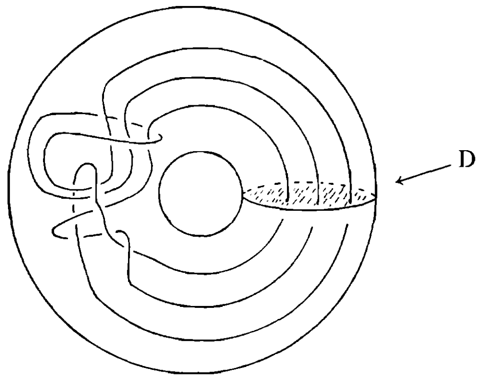

Figure 2.

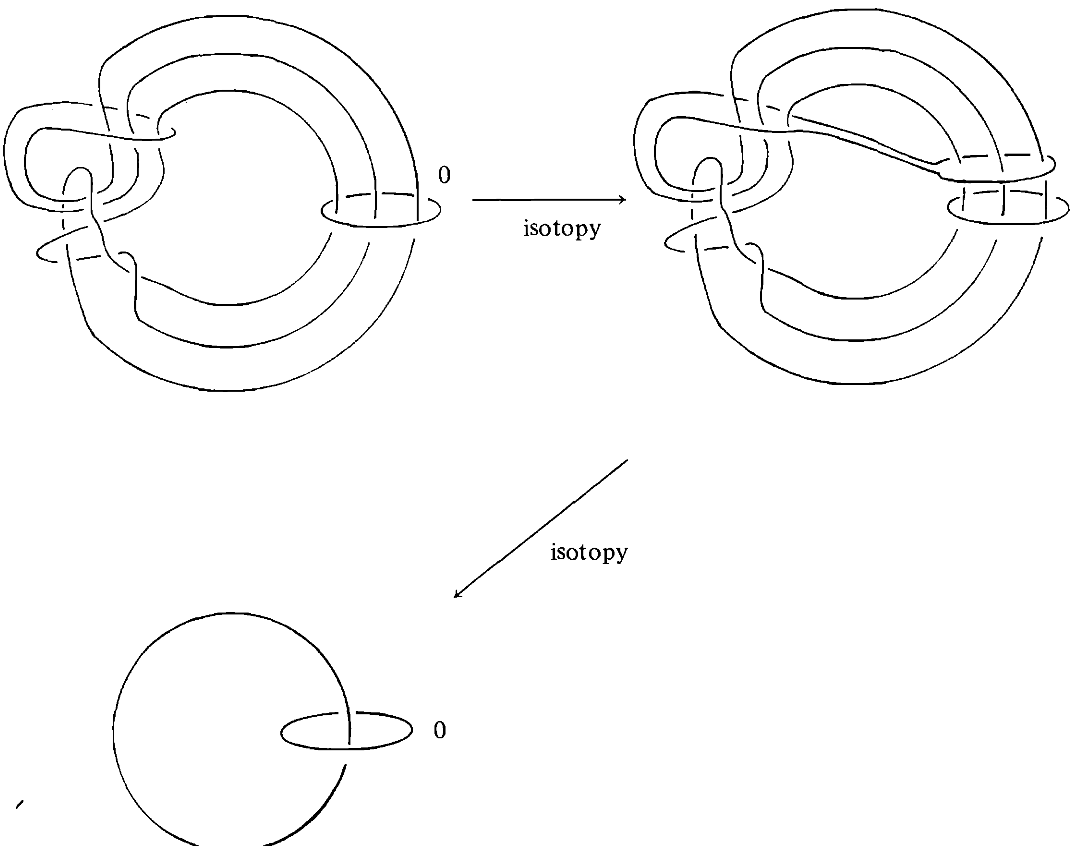

Figure 3.

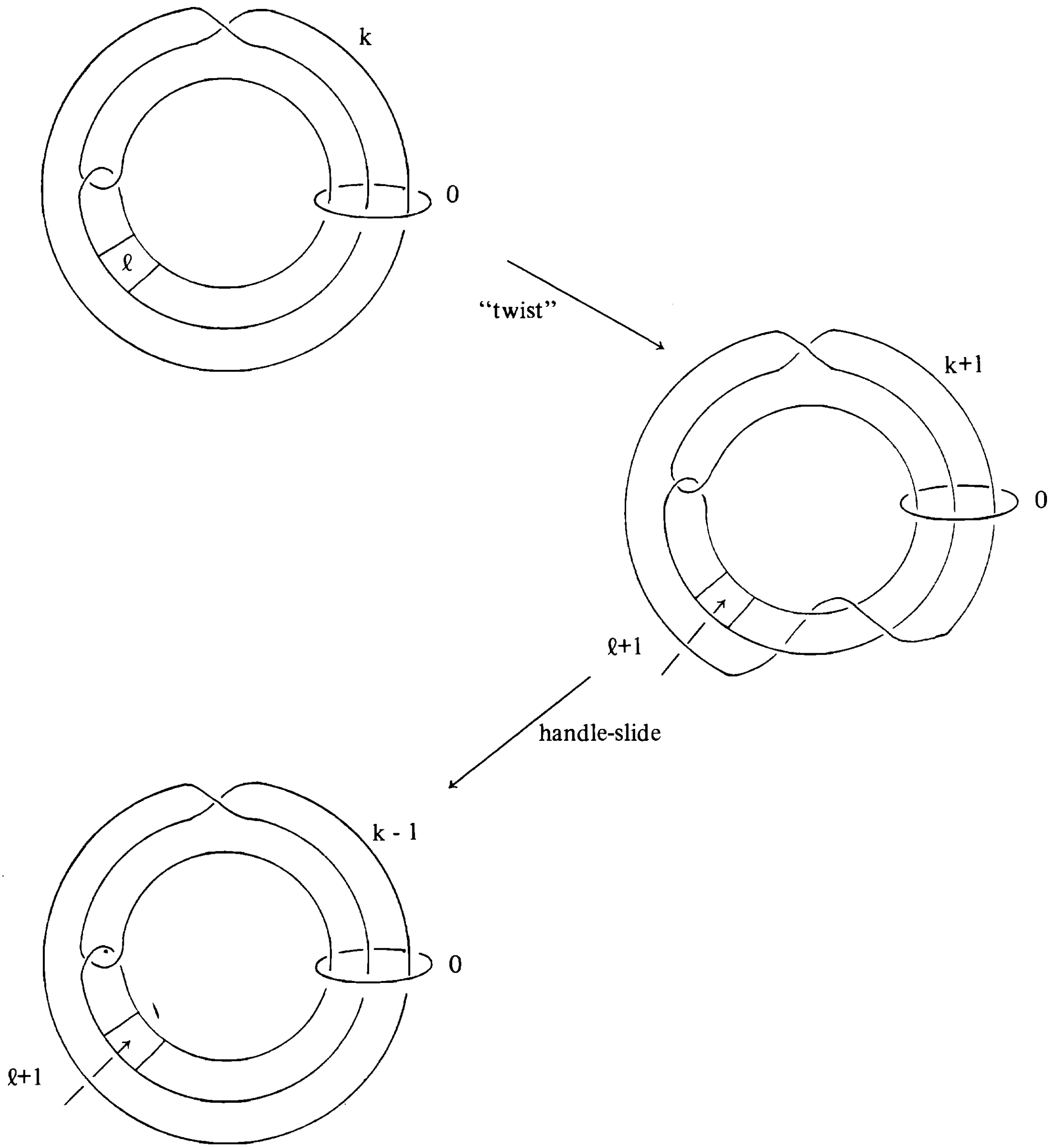

Figure 4.

There is an obvious 2-sphere S in $S^1 \times S^2$ intersecting K in 5 points. If some other 2-sphere T intersected K in fewer than 5 points we could assume T was disjoint from S, using the theorem of Section 2. Cutting $S^1 \times S^2$ along S results in $I \times S^2$, with K becoming 5 embedded arcs in $I \times S^2$. T is then an embedded essential 2-sphere in $I \times S^2$ intersecting at most 4 arcs. However, it is easy to see that 3 of the arcs run from $\{0\} \times S^2$ to $\{1\} \times S^2$ and hence T must intersect each of those 3 arcs. The remaining 2 arcs form a "clasp" in $I \times S^2$. Either an algebraic argument, computing the image of one arc in the relative homology group of the complement of the other arc or a geometric argument can be used to show those two arcs cannot be split by an embedded 2-sphere. Hence, T must intersect one of the arcs, and for simple algebraic reasons it must intersect that arc in two points.

Section 5. Equivalence of Mazur manifolds. In [2] it is shown that certain Mazur manifolds are diffeomorphic. Figure 4 illustrates that after a diffeomorphism of $S^1 \times B^3$ the attaching maps for the 2-handles of two such Mazur manifolds become isotopic. In the first illustration the attaching map, K, for the 2-handle of the Mazur manifold is drawn with framing as a curve in $S^1 \times S^2$. The generator of $\pi_1(SO(3))$ induces a diffeomorphism of $S^1 \times D^3$ to itself in the standard way. (This map has the effect of "twisting" $S^1 \times B^3$ once.) If this map is applied to $S^1 \times B^3$, K is carried to the curve in the second illustration, with framing changed as indicated. This curve is isotopic to the curve in the third illustration, again with indicated framing.

The equivalence of two Mazur manifolds just illustrated is one of the examples in [2]. The other examples can be constructed in a similar manner.

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Received June 15, 1982